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BRIDGES BETWEEN SUBRIEMANNIAN GEOMETRY AND ALGEBRAIC GEOMETRY: NOW AND THEN

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Abstract. We consider how the problem of determining normal forms for a specific class of nonholonomic systems leads to various interesting and concrete bridges between two apparently unrelated themes. Various ideas that traditionally pertain to the field of algebraic geometry emerge here organically in an attempt to elucidate the geometric structures underlying a large class of nonholonomic distributions known as Goursat constraints. Among our new results is a regularization theorem for curves stated and proved using tools exclusively from nonholonomic geometry, and a computation of topological invariants that answer a question on the global topology of our classifying space. Last but not least we present for the first time some experimental results connecting the discrete invariants of nonholonomic plane fields such as the RVT code and the Milnor number of complex plane algebraic curves.

1. Introduction.

One of the simplest idealized constraints one considers in nonholonomic mechanics is the *skate* or *no-slip* condition:

$$-\sin(\theta)dx + \cos(\theta)dy = 0. \tag{1}$$

where (x, y) are cartesian coordinates in the plane and $\hat{v} = (\cos(\theta), \sin(\theta))$ is the steering direction of the car – point along its axle. Stated plainly, the no-slip condition states that the wheels of the car are only allowed to roll along the road and, hence, no slipping occurs in the direction normal to the linear velocity of the car. Another closely related constraint is that which an airport luggage cart is subject to (Figure 1). In differential geometry such differential constraint is an example of a nonholonomic constraint, and it is well known that it does not possess integrable surfaces, i.e., it fails to satisfy the Frobenius condition of integrability ([1], Appendix 3).

We now present to the reader a series of mathematical miracles related to the contact distribution above, and her close relatives the so-called *Goursat distributions*.

To begin, consider the projectivization of $T\mathbb{R}^2$ which we will denote by $S(1)$. We denote $\mathbb{R}^2$ by $S(0)$. It is a simple exercise to show that $S(1)$ is diffeomorphic to $\mathbb{R}^2 \times S^1$. The projectivization lifts various objects canonically defined in $\mathbb{R}^2$, including the tautological

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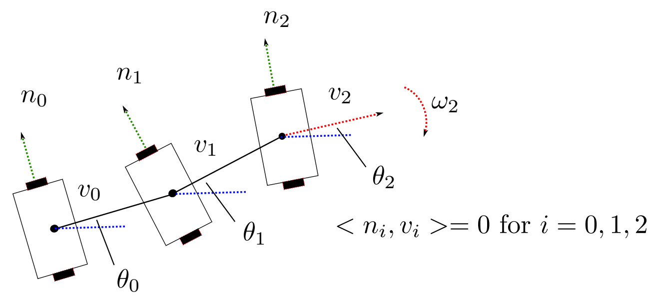
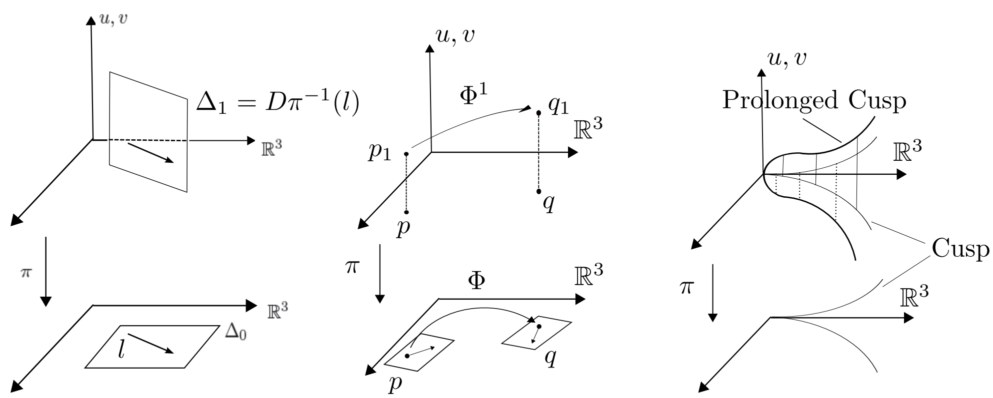


FIGURE 1. The car with trailers attached. The rate of change in the steering direction is denoted by ω_2.



(A) Prolongation of a distribution

(B) Prolongation of a diffeomorphism

(C) Prolongation of a curve

FIGURE 2. Prolongations

2-plane field, morphisms and curves. In Figure 2 we schematize the way various objects are lifted from $S(0)$ to $S(k)$.

We can coordinatize $S(1)$ by using projective coordinates on $\mathbb{R}^2$: $(x, y, [dx : dy])$ where the third entry represents the projective coordinates of a vector in $\mathbb{R}^2$. Let $\pi : S(1) \to S(0)$ denote the canonical projection of the bundle just defined. Fix $x \in S(0)$ and consider a line $l \subset T_x S(0)$. Then the plane field

$$\Delta_1(x, [l]) \doteq D\pi^{-1}(l).$$

The line l has implicit representation in the plane given by $-\sin(\theta)dx + \cos(\theta)dy = 0$. In local coordinates $[dx : dy] \cong \tan(\theta)$ and the inverse image in $S(1)$ of this line under the tangent map $D\pi$ is the plane spanned by the vectors $\left\{\frac{\partial}{\partial\theta}, \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}\right\}$. We have thus obtained the contact plane field (a.k.a skate constraint) from a geometric procedure known as **Cartan prolongation**. This *operation* can be iterated, and at each step we projectivize the plane field obtained in the previous step and the output is a tower of fiber bundles known as the Semple Tower ([8]):

$$\cdots \to S(n) \to S(n-1) \to \cdots \to S(2) \to S(1) \to S(0).$$

The Semple Tower has been rediscovered as the Monster Tower ([15],[16]). One can assign to each point a word in the letters R,V, and T, known as an *RVT code*, which is an invariant for the diffeomorphism group action. Each consecutive level $S(k)$ is endowed with a plane field Δ_k which is a Goursat Flag and a geometric model for the configuration space of a airport luggage convoy ([18], [15]). Some of these distributions had already been named within the literature of differential geometry; see [9].

One can generalize this construction to obtain a similar tower starting with a d-dimensional base manifold, but at each step one obtains a tower of fiber bundles with fibers isomorphic to $\mathbb{R}P^{d-1}$ (or S^{d-1} if orientation needs to be taken in into account). Each level of this tower will be also denoted by $S(k)$ without explicit mention to the base manifold. A word of caution here:

$$\dim(S(k)) = d + (d-1)k,$$

where d is the dimension of the base manifold. Correspondingly, each $S(n)$ is equipped with a d-dimensional plane field.

This tower can be thought of as the configuration space of a mechanical articulated arm [19]. There is also a mechanical resemblance between the articulated arm system just mentioned and the robotic snake model of Hausmann and Rodriguez [10], though we have to fix the position of the snake's tail. A simple geometric computation points out that for when the snake is completely stretched out the rank drops and it is not clear what the generic local behavior of this nonholonomic constraint is. Is this a Goursat distribution? Or a product constraint? By *product constraint*, we mean a plane field which contains factors as a product of a nonholonomic factor and trivial flat factor: $\Delta \times \mathbb{R}^k$. See section 5.3 of ([15]). Hausmann and Rodriguez determined the reachable sets for certain generic configurations of the snake, but have not discussed in detail the nature of the nonholonomic distributions ([10]) .

The multi-flags distributions will be one of the protagonists of our note, and by a theorem by Y. Shibuya and K. Yamaguchi ([21]) these generalized towers realize all of the so-called *Goursat multi-flags*.

> Our take home message to the reader is that serendipity is abundant when it comes to Goursat flags and multi-flags.

2. **Bridge 1: Nonholonomic geometry and curve singularities.** A first mathematical miracle in the study of Goursat flags or multi-flags is their connection with the singularity theory of smooth of analytic maps [2]. The fundamental notion underlying this connection is the *Cartan prolongation* already alluded to in the introduction. From the point of view of normal form theory, there is an action of the pseudogroup of local diffeomorphisms at the base $S(0)$. Different orbits will correspond to different normal forms. Given a point $p \in S(k)$ one associates to it $\Gamma(p)$ which is the set of all smooth curves through p that admit a nontrivial projection back to the base $S(0)$. It can be argued that $p \sim q$ ($\sim$ means equivalent under the group action) implies $\Gamma(p) \sim \Gamma(q)$. By symmetry, we fix the base point to be the origin and one can act on the fiber $\pi_k^{-1}(0)$, π_k being the canonical projection $S(k) \mapsto S(0)$. On this set of curves we act with the diffeomorphism group and using standard techniques of singularity theory of curves, different curve orbits (normal forms) will give rise to different normal forms of plane fields. The first successful step taken in this direction was documented in [16].

The task of labeling the orbits is sequential and it works by stages. In [5] we described in detail the general tools for this task for the Semple Tower with base $S(0) = \mathbb{R}^3$, though the construction and tools work for general base manifolds. There are two main approaches for determining the orbits:

1. the curve approach, and
2. the isotropy method.

Both methods are rather elementary and perform well in lower dimensions, depending mostly on the combinatorial or projective geometry of the problem at hand. Limitations to both methods are mostly computational in nature, since the amount of bookkeeping grows exponentially as one goes up the tower. Either approach is suitable to computer algebraic systems as pointed out in ([5]).

2.1. **Glimpses of algebraic geometry: Nash and Enriques.** Many features from classic algebraic geometry are easily transported to the Semple Tower if one thinks in terms of Nash blow-ups instead of quadratic transformations. For a modern reference on the subject see [13].

Cartan prolongations coincide with Nash blow-ups in the analytic category as explained in [3] and permitted us to make a case for the analogies between the problems in normal form theory for Goursat flags and some corresponding problems in enumerative geometry ([8]). A technical but rather important result in the classification problem of Goursat multi-flags consists of the following:

Theorem 2.1 ([3], Appendix B)**.** *Any well parametrized curve germ $c : I \to S(0)$ that is singular (i.e., $c'(0) = 0$) becomes regular (i.e. smooth and touching only regular points) after a finite number of Cartan prolongations.*

This theorem is crucial in defining the RVT code of a point in $S(k)$ in terms of Cartan prolongations of curves in the base $S(0)$. Let us use the notation $c^{(k)}$ for the k-th iteration of the Nash blow-up of a curve $c(t)$. Consider the list of points $\{c^{(k)}(0)\}$ obtained by evaluating the consecutive prolongations at $t = 0$. The proof consists of first showing that the set of points $c^{(k)}(0) \in S(k)$ contains only a finite number of critical points, and once we surpass the last critical point the curve become necessarily smooth. Otherwise it would have to be ill-parametrized. This is equivalent to the desingularization theorem stated in [13] but now formulated and proved using the language and tools of nonholonomic geometry.

As a corollary of the regularization theorem we obtain a nonholonomic version of the classical Enriques theorem about multiplicities of consecutive prolongations of a well-parametrized curve, though originally it was worded in terms of quadratic transformations and proximity relations. By *multiplicity* of a singularity we mean the first non-zero jet (if we mod out constants due to a specific choice of chart). This definition is suitable to parametrized curves and is independent of the coordinate chart by di Bruno's formula. We will exchangeably use the term multiplicity for either points or curves. Using special charts known as extended Kumpera-Ruiz coordinates (see [7]), the nested structure of Semple Towers (submanifolds of the original base manifold generate subtowers), and the regularization theorem above, we can prove the following:

Definition 2.2 ([3], Appendix B)**.** Let $p \in S(k)$. If a point q satisfies

- q is in the fiber above p, *or*
- q can be be reached by a prolongation of a vertical curve curve through p,

then we say that p and q are *adjacent points.* Points which are connected this way will form a graph (in fact, a tree) with seed p. The adjacency condition will be denoted by $q \to p$.

There is a simple relation between multiplies of adjacent points:

Theorem 2.3. *One has*

$$mult(p) = \sum_{q \to p} mult(q).$$

This a classic result in enumerative geometry attributed to F. Enriques, and in our context it restricts the class of singular points that be reached from a given point via Cartan prolongation. The proof is again based purely on the local geometry of the nonholonomic fields in the Semple tower.

2.2. **Puiseux numbers and growth vectors.** In [20], we gave a formula for the Puiseux characteristic of an analytic plane curve germ which represents a Goursat distribution germ with prescribed small growth vector.

Given a Goursat distribution D on a manifold M, consider the sequence $D_i = [D, D_{i-1}] + D_{i-1}$, where $D_0 = D$. Then there exists an r such that $D_r = TM$. For each $p \in M$, we define the *small growth vector at* p to be the integer valued vector

$$sgv(p) = \big(\dim D_0(p),\ \dim D_1(p), \dots,\ \dim D_r(p) = n\big).$$

The *derived vector* of a Goursat germ consists of the multiplicities of the entries in the small growth vector. For a Goursat distribution, the dimensions of the sequence D_i grow by at most one at a time, so the multiplicities are nonzero and from the list of multiplicities we may recover the original small growth vector.

For a well-parametrized, non-immersed plane curve

$$\gamma(t) = (t^m, \sum_{k \geq m} a_k t^k)$$

the Puiseux characteristic is defined as follows. Let $\lambda_0 = e_0 = m$. Then define inductively

$$\lambda_{j+1} = \min\{k \mid a_k \neq 0,\ e_j \nmid k\}, \qquad e_{j+1} = \gcd(e_j, \lambda_{j+1})$$

until we first obtain a g with $e_g = 1$. Then the vector $[\lambda_0; \lambda_1, \dots, \lambda_g]$ is called the *Puiseux characteristic* of γ. The Puiseux characteristic is the fundamental invariant in the singularity theory of plane curves. In [23], Proposition 4.3.8 shows that it is equivalent to at least seven other classical invariants.

In short, [20] provided the dashed arrow in the following diagram:

$$\begin{tikzcd}
\{SGV\} \arrow[r, shift left] \arrow[dr, dashed] & \{RVT\} \arrow[l, shift left] \arrow[d, leftrightarrow] \\
& \{PC\}
\end{tikzcd}$$

Here, PC represents the Puisuex characteristic of a plane curve, RVT represents the RVT code of a point in the Monster Tower, and SGV represents the small growth vector of a Goursat germ. The arrow $\{\text{RVT}\} \longrightarrow \{\text{SGV}\}$ was given in [11], the arrow $\{\text{RVT}\} \longleftrightarrow \{\text{PC}\}$ was given in [16], and the arrow $\{\text{SGV}\} \longrightarrow \{\text{RVT}\}$ was given in [17].

Now suppose we are given a Goursat germ whose derived vector is

$$der = (\underbrace{M_1,\ M_1, \dots, M_1}_{m_1},\ \underbrace{M_2,\ M_2, \dots, M_2}_{m_2}, \dots, \underbrace{M_{v+1},\ M_{v+1}, \dots, M_{v+1}}_{m_{v+1}}),$$

with $M_1 < M_2 < \cdots < M_v < M_{v+1}$. Consider the set $S = \{M_i |\ M_{i-1} \text{ divides } M_i\}$. Let $g = |S|$. For $1 \leq j \leq g$, let $N_1, N_2, \dots, N_g$ denote the elements of S in decreasing order. We always have $N_g = M_2$, since $M_1 = 1$. For $1 \leq j \leq g$ let $M_{k_j} = N_j$.

Theorem 2.4 ([20])**.** *The corresponding Puiseux characteristic is* $[\lambda_0; \lambda_1, \dots, \lambda_g]$ *where*

$$\begin{aligned}
\lambda_0 &= M_{v+1} \\
\lambda_j &= \sum_{i \geq k_j} m_i M_i + M_{k_j} + M_{k_j - 1}
\end{aligned}$$

for $1 \leq j \leq g$.

Example 1. Suppose $der = (1, 1, 2, 2, 2, 2, 2, 2, 4, 6, 6, 6, 18, 24, 24)$. Note that $\lambda_0 = M_{v+1} = M_6 = 24$. We also have $S = \{18, 4, 2\}$, and therefore $g = 3$. Then write $S = \{18, 4, 2\} = \{N_1, N_2, N_3\} = \{M_5, M_3, M_2\}$ so that $k_1 = 5$, $k_2 = 3$, and $k_3 = 2$. Finally, we compute

$$\lambda_1 = \sum_{i\geq 5} m_i M_i + M_5 + M_4 = 90$$

$$\lambda_2 = \sum_{i\geq 3} m_i M_i + M_3 + M_2 = 94$$

$$\lambda_3 = \sum_{i\geq 2} m_i M_i + M_2 + M_1 = 103.$$

The Puiseux characteristic is thus $[24; 90, 94, 103]$.

2.3. **Spelling rules.** The RVT code was first studied for the $\mathbb{R}^2$-Semple Tower and is a word in the letters R, V, and T subject to a simple set of spelling rules ([16]). The spelling rules come from the number of critical directions that appear in the rank 2 distribution that exist above each point in the planar tower. In [5], Montgomery, Howard and Castro began studying the $\mathbb{R}^3$-Semple Tower and extended the alphabet for the RVT coding system to include the letters T_i for $i = 1, 2$ and L_j for $j = 1, 2, 3$ which come from the critical planes that exist within the rank 3 distributions at each level of the $\mathbb{R}^3$-Semple Tower. The first spelling rules were obtained in [7], [4], and in [6] we investigated the behavior of these critical planes and completed the spelling rules. These spelling rules for the spatial tower are given by the following result, where the ":" denotes which letters can be placed after a given letter. For example, given the letter R one can put either the letters R or V after it.

Theorem 2.5. [6] *The complete spelling rules for any RVT code in the $\mathbb{R}^3$-Semple Tower are as follows:*
(1) *Any RVT code string must begin with the letter R.*
(2) R : *R and V.*
(3) *V and $T(= T_1)$: R, V, T, and L.*
(4) *$L(= L_1)$ and L_j for $j = 2, 3$: R, V, T_i for $i = 1, 2$, and L_j for $j = 1, 2, 3$.*
(5) *T_2 : R, V, T_2, and L_3.*

The significance of this result is the role it plays in the classification problem of the points within the spatial tower. In [7] we used a technique called the *isotropy method* which allows us to classify points at any level of the spatial tower so long as we know how to describe the RVT-classes in Kumpera-Ruiz coordinates.

Theorem 2.6 ([4])**.** *In the spatial Semple Tower the number of orbits within each of the first four levels of the tower are as follows:*
- *Level 1 has 1 orbit,*
- *Level 2 has 2 orbits,*
- *Level 3 has 7 orbits,*
- *Level 4 has 34 orbits.*

3. **Bridge 2: Nonholonomic geometry and algebraic topology.**

3.1. **Semple meets Chern: Nontrivality of the $\mathbb{C}^2$-Semple Tower.** Some interesting algebraic topological questions arise when one starts to consider the complexified version of the Semple Tower. If we replace $S(0)$ by $\mathbb{C}^2$, the fibers over the origin cease to be a trivial Cartesian product as the following computation with cohomology classes show.

We will show, for a base consisting of a neighborhood U of the origin in $\mathbb{C}^2$ that the nth level of the Semple Tower is *not* the product manifold $U \times (\mathbb{C}P^1)^n = U \times \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$. We can show this by using the *Borel-Hirzebruch formula*, found in [12], in order to compute the cohomology of the $\mathbb{C}^2$-Semple Tower.

Let ξ be a rank n complex vector bundle over a topological space X, and let $\mathbb{P}(\xi)$ denote its projectivization. Then the *Borel-Hirzebruch formula* is given by

$$H^*(\mathbb{P}(\xi);\mathbb{Z}) \simeq H^*(X;\mathbb{Z})[x] / < x^n + \sum_{i=1}^{n}(-1)^i c_i(\pi^*\xi)x^{n-i}>,$$

where $\pi^*\xi$ is the pullback of ξ along $\pi : \mathbb{P}(\xi) \to X$ and $c_i(\pi^*\xi)$ if the ith Chern class of $\pi^*\xi$. Here, x can be viewed as the first Chern class of the canonical line bundle over $\mathbb{P}(\xi)$. One can also replace $c_i(\pi^*\xi)$ with $c_i(\xi)$ since the induced homomorphism $\pi^* : H^*(X;\mathbb{Z}) \to H^*(\mathbb{P}(\xi);\mathbb{Z})$ is injective. We can apply the Borel-Hirzebruch formula to an n-level $\mathbb{C}P$-tower

$$S(m) \xrightarrow{\pi_m} S(m-1) \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} S(1) \xrightarrow{\pi_1} S(0) = \{\text{a point}\},$$

with $S(i) = \mathbb{P}(\xi_{i-1})$, to get the isomorphism

$$H^*(S(m);\mathbb{Z}) \simeq \mathbb{Z}[x_1,\cdots,x_m]/ < x_k^{n_k} + \sum_{i=1}^{n_k}(-1)^i c_i(\xi_{k-1})x_k^{n_k-i} \,|\, k = 1,\cdots,m >$$

Theorem 3.1. *For $n \geq 2$, the nth level of the $\mathbb{C}^2$-Semple Tower is a nontrivial bundle and the cohomology at each level of the tower is of the form $H^*(S(n);\mathbb{Z}) \simeq \mathbb{Z}[x_1,\cdots,x_n]/ < x_1^2, x_k^2 - c_1(\Delta_{k-1})x_k \,|\, k = 2,\cdots,n >$.*

Proof. The first level of the Semple Tower is a trivial bundle given by $S(1) = U \times \mathbb{C}P^1$ with U being a contractible open subset of the origin in $\mathbb{C}^2$. Our rank 2 distribution over $S(1)$ is $\Delta_{(p,\ell)} = d\pi^{-1}_{(p,\ell)}(\ell)$ for $p \in U$ and $\ell \subset T_p\mathbb{C}^2$. We use the approach given in [14] to determine the first and second Chern classes for Δ_1. We note that there is a nonvanishing section $s : S(1) \to \Delta_1$ given by $(p,\ell) \mapsto \ell$, since ℓ is never zero and hence tells us that the second Chern class of Δ_1 will vanish. Let Δ_1^0 be the rank 1 subdistribution of Δ_1 defined by $\Delta_1^0(p,\ell) = \Delta_1(p,\ell)/span\{\ell\}$ which will be the tangent space to $\mathbb{C}P^1$. It is well know that $T\mathbb{C}P^1$ has nontrivial first Chern class. This implies $c_2(\Delta_1) = 0$ and $c_1(\Delta_1) \neq 0$, and since $S(2) = \mathbb{P}(\Delta_1)$ we end up with $H^*(S(2);\mathbb{Z}) \simeq \mathbb{Z}[x_1,x_2]/ < x_1^2, x_2^2 - c_1(\Delta_1)x_2 >$. One can see that we can apply the same reasoning as above to show $c_2(\Delta_i) = 0$ and $c_1(\Delta_i) \neq 0$ for $i \geq 2$ and that $H^*(S(n);\mathbb{Z}) \simeq \mathbb{Z}[x_1,\cdots,x_n]/ < x_1^2, x_i^2 - c_1(\Delta_{i-1})x_i \,|\, i = 2,\cdots,n >$. □

The main open question here is: *Can one realize these cohomology classes as singularity classes within the Semple Tower?* R. Thom, who to our knowledge, was the first to propose this sort of program in algebraic topology ([22]). Whether this realization has direct applications in controllability or stabilization questions of the underlying control system remains elusive to us.

3.2. **Semple meets Milnor.** A parallel definition of the RVT codes for Goursat germs was proposed in [16] using the Semple Tower. This tower is Goursat universal: every Goursat germ occurs somewhere within the tower. Each point in the Semple Tower is assigned an RVT code, and the code of a Goursat germ at a reference point p is that of p itself. See [16] for details, or Section 2.2 of [20] for a summary.

In [16], a correspondence between points in the Semple Tower and plane curve germs was made explicit. Singular curves correspond to points whose RVT code ends with the letter V or T. The Milnor number is a fundamental invariant of such curve singularities. Our current work seeks to compute the Milnor number μ from a given RVT code, and we present some preliminary results below, after recalling the definition of μ.

Suppose C is the germ at O of the singular plane curve defined by $f(x,y) = 0$. Let B_ϵ denote the disk of radius ϵ centered at the origin in $\mathbb{C}^2$, with boundary sphere S_ϵ.

Definition 3.2. Let $K = f^{-1}(0) \cap S_\epsilon$. Then for sufficiently small ϵ, the map

$$\begin{aligned} \phi \colon S_\epsilon - K &\to S^1 \\ z &\mapsto f(z)/|f(z)| \end{aligned}$$

is a fibration, known as the *Milnor fibration.*

The fiber F, known as the *Milnor fiber*, is a compact, connected, oriented surface with r boundary components, where the curve C has r branches. The first Betti number of F is the *Milnor number* of the singularity, denoted μ.

The two formulas below are conjectured to give the Milnor number for a prescribed RVT code. The first formula (*) concerns a single block of the form $R^sV^kT^u$ where the parameters s, k, and u are arbitrary non-negative integers. Any RVT code consists of a sequence of such blocks. Here the superscripts denote multiplicities of letters. The second formula (**) concerns RVT codes which consist of strings of the form $R^{s_j}VT^{u_j}$, where the parameters s_j and u_j are positive integers. Proofs of these formulas will appear in a forthcoming paper of Howard and Shanbrom. Here $F(k)$ denotes the kth Fibonacci number, where $F(1) = F(2) = 1$. Also, we consider $\mu/2$ instead of the Milnor number μ for convenience, and $\binom{k}{2}$ denotes k choose 2.

Basic building block:

$$\begin{aligned} \frac{\mu}{2}(R^sV^kT^u) = \binom{F(k+2)}{2}(2+2u) - \binom{F(k+1)}{2}u + \binom{F(k+2)+F(k)u}{2}(s-2) \\ + F(k)F(k+2)\frac{u(u+1)}{2} + \sum_{j=1}^{k-1}\binom{F(j+2)}{2}. \end{aligned} \tag{*}$$

For example, $\mu(R^3V^5T^2) = 1804$.

Iterative process for single V's:

$$\begin{aligned} \frac{\mu}{2}(R^{s_1}VT^{u_1}R^{s_2}VT^{u_2}\cdots R^{s_n}VT^{u_n}) = &\binom{(u_1+2)\cdots(u_n+2)}{2}s_1 \\ &+ \binom{(u_2+2)\cdots(u_n+2)}{2}(s_2+u_1+1) + \dots \\ &+ \binom{(u_j+2)\cdots(u_n+2)}{2}(s_j+u_{j-1}+1) + \dots \\ &+ \binom{(u_n+2)}{2}(s_n+u_{n-1}+1). \end{aligned} \tag{**}$$

For example, $\mu(R^2VT^2R^3VTRVT^3) = 8400$.

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